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CHAOTIC MOTION IN A NONLINEAR OSCILLATOR WITH FRICTION

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The mechanical oscillator with dry friction, Duffing type nonlinearity and exciting periodic force is investigated. An analytical approach is presented for finding the critical parameter sets for the Hopf bifurcation of the previous stationary state. The numerical simulations has proved the occurrence of the strange attractor near the critical values of the parameters.

Key Words : Chaos, Strange Attractor, Hopf Bifurcation, Bifurcation Equations, Fast Fourier Transform

1. INTRODUCTION

Beginning with the decisive works by Lorenz and Ueda it appears that even in simple discrete nonlinear systems at least of the third order for some parameter sets chaotic orbits appear and the set which attracted them is called strange attractor (Lorenz, 1963 ; Ueda, 1979). To date there are many examples of chaotic motion in simple physical systems (Troger, 1982 ; Szemplinska-Stupnicka, 1987 Awrejcewicz, 1986a, b, c ; Kapitaniak, Awrejcewicz, Steeb, 1987). The study of nonlinear dynamics in these simpler systems provides the advantage of less complex analysis of the behaviour of strange attractors and, additionally, these or similar phenomena can also appear in higher dimensional systems analogues.

For the purpose of introducing the reader to the question we will first present some basic ideas connected with the field of chaotic dynamics of dissipative systems, which is now undergoing explosive growth.

Consider the system of ordinary differential equations

$$\dot{x}=F_\mu(x), \tag{1}$$

with $x\in R^n$, and $\mu\in R^k$. We assume that F_μ is smooth and depends smoothly on μ. Let the curves $\phi_\mu(x,\ t)$ be the solutions of (1) with initial conditions x. A bifurcation we define as a transition from one topological equivalence class to another. Let us suppose that the system (1) has an equilibrium path $e(\mu_0)$ with the parameter value μ_0 at which the Jacobian DF_{μ_0} has a simple pair of pure imaginary eigenvalues $\lambda_c=\pm iw_0$ and also $w_0>0$ and $d/d\mu(Re(\lambda(\mu))_{\mu=\mu_0}\neq 0$. Moreover all the other eigenvalues possess negative real parts. There is a theorem given by Hopf which describes the qualitative properties of the flow near this critical value of the parameter μ(Hope, 1942). In this case, when a pair of complex conjugate eigenvalues cross from the left-hand plane into the right-hand plane, a stable periodic orbit is born, and previous stable equilibrium becomes unstable. This phenomenon is widely described in the literature (see for instance Marsden and McCracken, 1976 ; Hassard, Kazarinoff and Wan, 1980 ; Iooss and Joseph, 1981 ; Holmes, 1981 ; Guckenheimer, 1983).

Assume now that the system (1) has a periodic orbit α for the flow ϕ_{μ_0}. This flow we can study in the neighbourhood of μ_0 and α using the return map $\Gamma:\xi\rightarrow\xi$, where surface ξ crosses α transversally. Let α be a periodic point of period q i.e. $\Gamma^q(\alpha)=\alpha$ and $\Gamma^n(\alpha)\neq\alpha$ for $0<n<q$. For the map, a periodic orbit associated with α is the set

$$\alpha_0=\{\alpha,\ \Gamma(\alpha),\ \Gamma^2(\alpha),\ \cdots,\ \Gamma^{q-1}(\alpha)\}. \tag{2}$$

If $q=1$ we have a periodic orbit with the period 1 and in this case the map has a fixed point α_0. The Jacobian $D\Gamma(\alpha_0)$ gives information about the flow near the periodic orbit α. If every eigenvalue of $D\Gamma(\alpha_0)$ possesses an absolute value smaller than 1 then the periodic orbit is stable in the Liapunov sense and all trajectories in the neighbourhood of α approach it as $t\rightarrow\infty$. If at least one of the eigenvalues has an absolute value greater then one, the considered periodic orbit is unstable. In this case bifurcation of the periodic orbit appears (secondary type bifurcation). Let us suppose that a pair of complex eigenvalues of $D\Gamma(\alpha)$ cross the unit circle at $\mu=\mu_1$. The flow ϕ_μ for $\mu>\mu_1$ has an invariant torus which contains periodic or quasiperiodic orbits. But in some cases with the further changing of the parameter μ the attractor grows and starts to warp, eventually becoming the strange attractor (Aronson, et. al., 1982).

As Ruelle suggests however, there is as yet no completely satisfactory mathematical definition of the strange attractor which is universally accepted (Ruelle, 1980). For our purpose we will say that the considered dynamical system governed by equations set (1) has a strange attractor if there is an orbit which does not appear to converge to classic attractors as a fixed (equilibrium) point, a periodic orbit or quasi periodic orbit. Additionaly, the strange attractor has certain properties which allows us to investigate this new dynamical phenomenon.

Grebogi et. al. distinguish the strange attractors which are

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chaotic and non-chaotic. The word strange refers to the geometry or shape of the attracting set, while the word chaotic refers to the dynamics of orbits on the attractor. The dimensions of the classical attractors are integer, (the equilibrium point has zero dimension; the limit cycle has one dimension), while the dimension of the strange attractor is not integer but fractal. Additionaly, the trajectories belonging to the strange chaotic attractor have the property known as "sensitive dependence on initial conditions" and the power spectra of dynamical variables are broadband. For the chaotic orbits the distance between two points that were initially very close to each other grows exponentially fast on the average. The largest Liapunov exponent describes this rate of separation between two orbits, and if this exponent is positive we define this attractor as a strange chaotic.

For the complicated dynamical systems there are no generally efficient analytical methods of solving the adequate equations set and numerical methods based on the initial value problems are generally poor. However, one can avoid these problems by using recurrence map consists in the reduction of the dimensionality of the problem by one unit, which is accomplished by the elimination of one variable via a suitable "surface of section" (Gumowski, Mira, 1980). For the periodically excited systems in an n-state variable problem it is very convenient to use a poincaré section. In this case a natural samling rule is to choose time $t_n = nT + \tau_0$ and then to calculate the discrete values of the dynamic variables at this discrete time, where T is a period of exciting force (Moon, 1987). For our purposes we then use the Poincaré section to discover the strange chaotic attractor in the considered forced oscillator with friction.

Strange attractors have been usually sought in a random way and it has been difficult to present a general method to discover them. On the other hand, the transition to chaos in the forced oscillators is connected with a sequence of successive bifurcations which precede this irregular motion. Ruelle and Takens were the first to suggest that strange attractors could arise after a finite sequence of bifurcations and might provide models for complicated ones (Ruelle, Takens, 1971). The scheme which is given by Ruelle and Takens shows how, after three, two or even one Hopf bifurcations, the system can undergo a subsequent transition to chaos.

In this paper it will be shown how chaos can arise after the Hopf bifurcation of the stationary state with one frequency.

2. THE ANALYSED SYSTEM, HOPF BIFURCATION FOR $\nu A > 1$ AND NUMERICAL RESULTS

The equation of motion for the analysed oscillator is:

$$m\ddot{x} + k_0 x + k_1 x^3 = mg(\mu_0 \mathrm{sign}(v_0 - \dot{x}) - (v_0 - \dot{x}) + \beta(v_0 - \dot{x})^3) + P_0 \cos \omega t, \tag{3}$$

where m is mass of a vibrating body, k_0 and k_1 are stiffness coefficients, μ_0, α and β are friction coefficients, $P_0$and ω are adequate amplitude and frequency of exciting force and ν_0 is the velocity of the tape, on which lies the vibrating body. As is well known in such autonomous systems the self-excited vibrations can appear, which are caused by friction. The situation, however, is more complicated when a system is excited by harmonic force. We consider two cases with regard to the argument of sign function, because in both cases we will obtain the different bifurcation equations after using an averaging procedure. The case $v_0 = \dot{x}$ means that the body with mass m does not move with respect to the tape.

From Eq. (1) we get:

$$y'' + y + ay^3 = b\mathrm{sign}(1 - y') - e(1 - y') + h(1 - y')^3 + q\cos \nu\tau, \tag{4}$$

where:

$$y = (k_0/m)^{1/2} x v_0^{-1}, \quad b = \mu_0 g v_0^{-1} (k_0/m)^{-1/2},$$
$$\tau = (k_0/m)^{1/2} t, \quad e = g\alpha (k_0/m)^{-1/2},$$
$$\nu = (m/k_0)^{1/2} w, \quad h = \beta g v_0^2 (k_0/m)^{-1/2},$$
$$a = mk_1 v_0^2 k_0^{-2}, \quad q = P_0 m^{-1} v_0^{-1} (k_0/m)^{-1}. \tag{5}$$

The operator ($\dot{}$) means differentiation with regard to t, and (') with regard to τ. Supposing that the system has a solution

$$y = A_0 + A_1 \cos \nu\tau + B_1 \sin \nu\tau, \tag{6}$$

and that $\nu A > 1$ (where $A = (A_1^2 + B_1^2)^{1/2}$) the Eq. (2) gives

$$KA_1 + LB_1 - q + 4b(1 - (\nu A)^{-1/2})^{1/2\pi^{-1}} = 0,$$
$$KB_1 - LA_1 = 0,$$
$$A_0\left(1 + a\left(A_0^2 + \frac{3}{2}A^2\right)\right) - h\left(1 + \frac{3}{2}\nu^2 A^2\right) + e - + b\left(2 - \frac{4}{\pi}\mathrm{arc}\cos(\nu A)^{-1}\right) = 0,$$
$$K = 1 - \nu^2 + 3aA_0^2 + \frac{3}{4}aA^2,$$
$$L = \nu(3h - e) + \frac{3}{4}h\nu^3 A^2. \tag{7}$$

Consider the perturbated solution

$$y_p = A_0 + (A_1 + \xi(\tau)\cos \nu\tau + (B_1 + \eta(\tau))\sin \nu\tau, \tag{8}$$

and taking into account the assumption that the coefficients b, e, h are small and ξ' and η' are slowly changing function of τ from Eq. (4) one can obtain:

$$-2\nu\xi' + M\xi + N\eta = 0,$$
$$2\nu\eta' + P\xi + R\eta = 0, \tag{9}$$

where:

$$M = \frac{3}{2}A_1 B_1 a + e\nu - 3h\nu - \frac{9}{4}h\nu^3 A_1^2 - \frac{3}{4}h\nu^3 B_1^2,$$
$$N = 1 - \nu^2 + 3A_0^2 a + \frac{9}{4}B_1^2 a + \frac{3}{4}A_1^2 a - \frac{3}{2}A_1 B_1 \nu^3 h,$$
$$P = 1 - \nu^2 + 3A_0^2 a + \frac{9}{4}A_1^2 a + \frac{3}{4}B_1^2 a + \frac{3}{2}h\nu^3 A_1 B_1 + + \frac{4bA_1}{\pi\nu^2 A^4}\left(1 - \frac{1}{\nu^2 A^2}\right)^{-1/2},$$
$$R = \frac{3}{2}A_1 B_1 a - e\nu + 3\nu h + \frac{3}{4}h\nu^3 A_1^2 + \frac{9}{4}h\nu^3 B_1^2 + + \frac{4bB^1}{\pi\nu^2 A^4}\left(1 - \frac{1}{\nu^2 A^2}\right)^{-1/2}. \tag{10}$$

The necessary condition for Hopf bifurcation to appear [see for instance (Arrowsmith, Taha, 1983)] leads to:

$$R - M = 0, \tag{11}$$

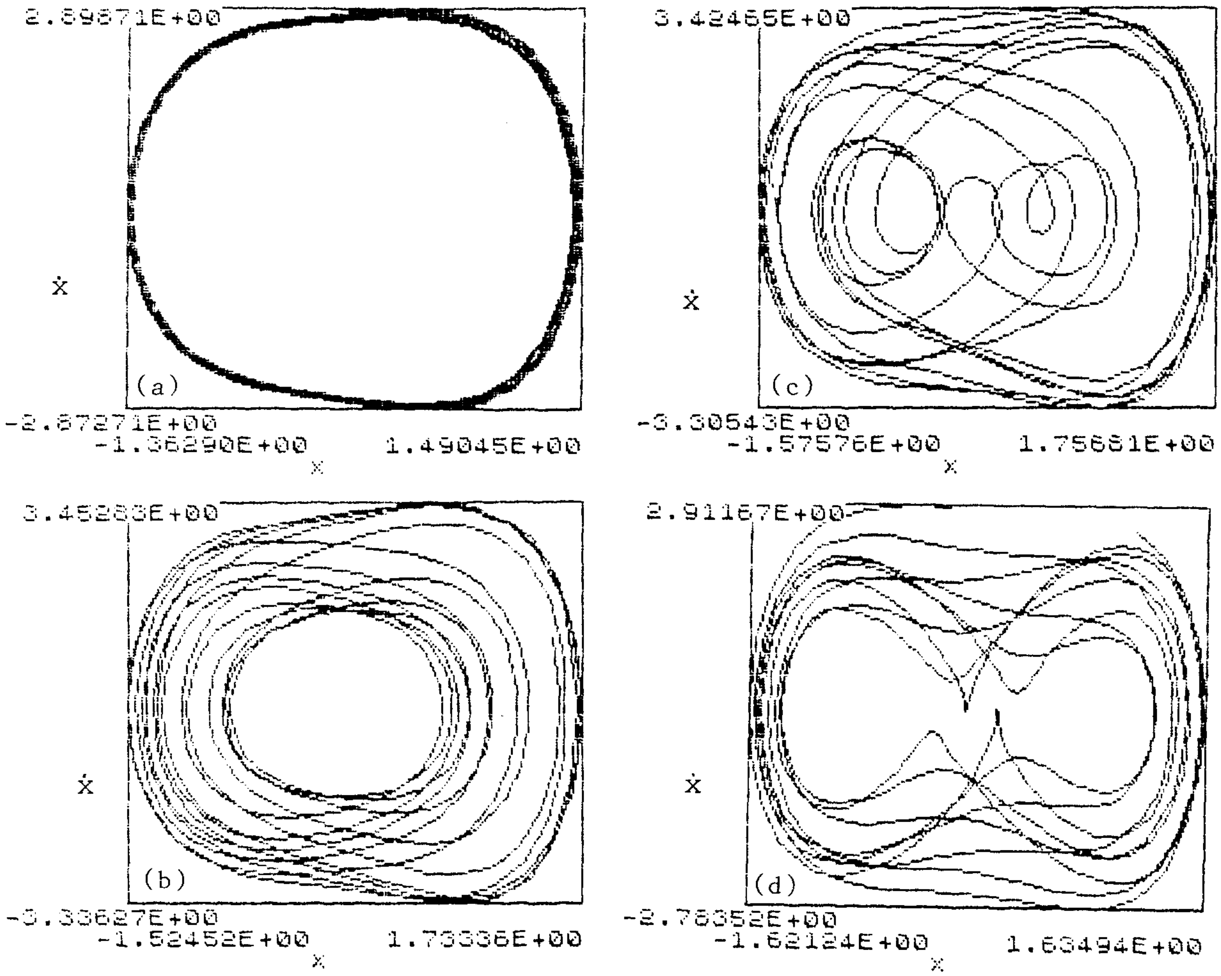

Fig. 1 Phase portraits of the analysed system: $a=3.$, $h=.05$, $e=.5$, $b=.08$, $\nu=1.4$. (a) $q=.1$; (b) $q=1.5$; (c) $q=2.5$; (d) $q=4$.

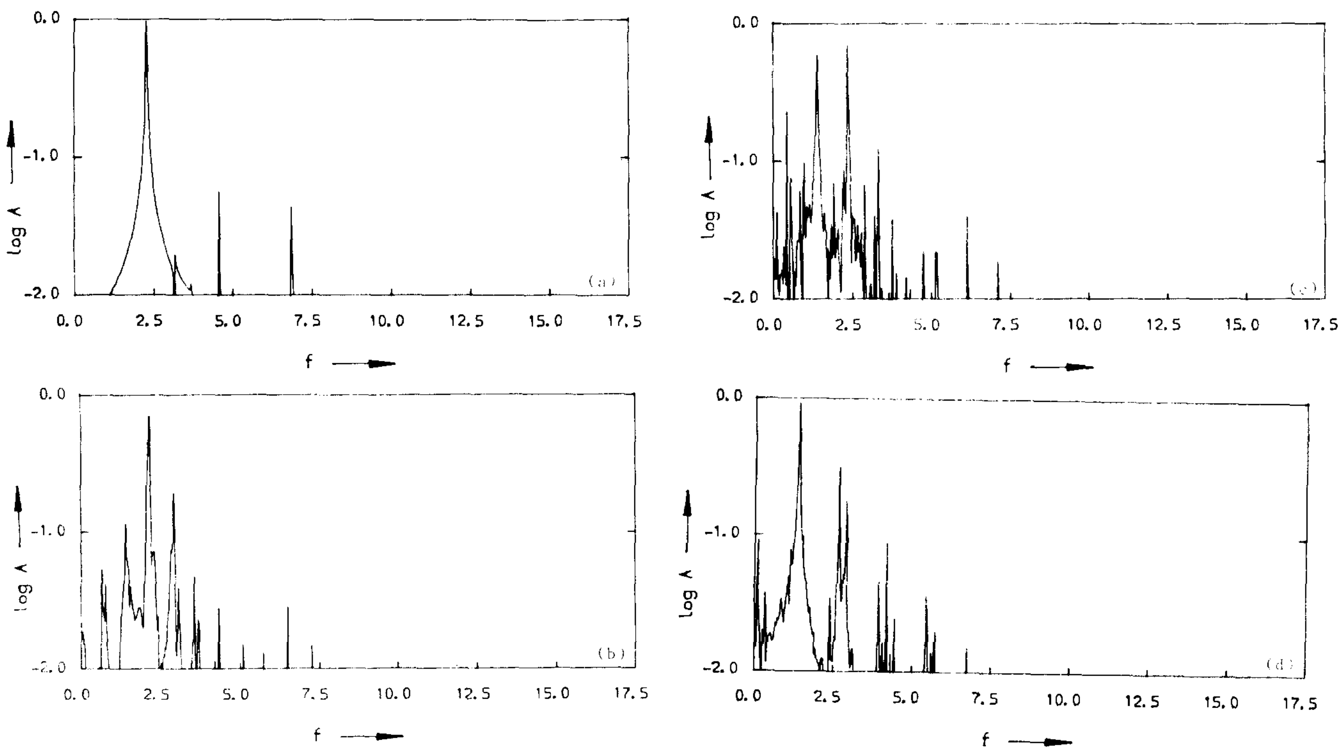

Fig. 2 Frequency spectra obtained using Fast Fourier Transform for the parameters as in Fig. 1.

$$PN - MR > 0. \tag{12}$$

The Eq. (9) and Eq. (11) are the bifurcation equations. The set of parameters which satisfy these equations and the inequality (12) is a bifurcation vector. In order to obtain this vector, a, ν, h, e, and b are fixed for each step of iteration and then the Monte Carlo method with lottery-drawing generator of number was used to solve the bifurcation equations in regard to unknown parameters A_0, A_1, B_1 and q. Only the isolated solution was found succesfully $(a, \nu, h, e, b, q) = (1., 1.5, .05, .5, .58)$ which satisfy inequality (12).

The Eq. (4) was solved numerically using modified Runge-Kutta method for the parameters lying nearby the bifurcation vectors. All calculations were made with the initial conditions $y(0) = y'(0) = 0$, and the phase portraits were recorded from time $T_{min} = 80$ to $T_{max} = 120$ wehere the trajectory has reached finally attractor. The question arises, however, whether with the changing of the initial conditions the trajectories can reach another stable attractor. Examples of this kind of coexistance of a "strange attractor" and a large periodic orbit were given by Holmes and Stupnicka-Szemplinska (Holmes, 1979 ; Stupnicka-Szemplinska, 1987). In these cases, for some initial conditions the trajectory reaches a strange chaotic attractor, whereas for the others a stable limit cycle. In our example we have changed the initial conditions in the neighbourhood of the point (0, 0) but we have not detected another attractor.

Now we present using phase portraits, the characteristic behaviour of the evolution of attractors firstly with the change in parameters a, b and then secondly with the amplitude of dimensionless exciting force q. In Fig. 1(a) for $q = 0.1$ we have obtained a periodic orbit. With a further increase of this parameter value the trajectories start intersecting each other [Fig. 1(b)] and then for $q = 2.5$ chaos has appeared.

The characteristic picture of the strange chaotic attractor is shown in Fig. 1(c). The corresponding Fourier spectra are presented in Fig. 2. The most irregular spectrum is obtained corresponding to the $q = 2.5$. It resembles a broad band and testifies that the motion is chaotic. For regular motion, the Fourier spectra possess only descrete values. We have also presented the view of a strange chaotic attractor using the Poincaré section (the method of construction was described in sec. 1). For the parameters (3., 1.5, .05, .5, .08, 2.5) the strange chaotic attractor is presented in Fig. 3.

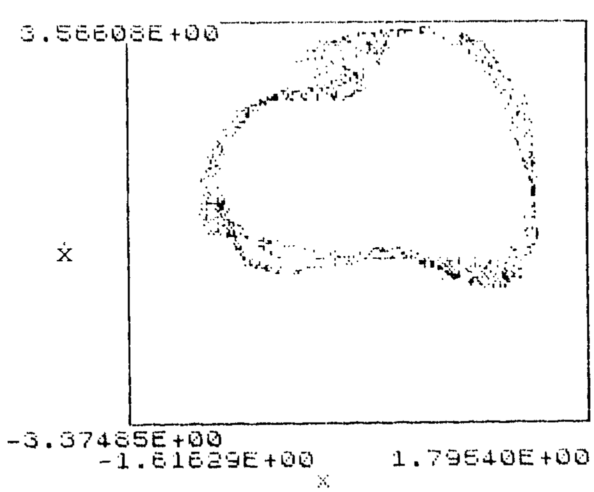

Fig. 3 Poincaré map for the parameters of the Fig. 1(c).

3. HOPF BIFURCATION FOR $\nu A < 1$ AND NUMERICAL RESULTS

Doing similar as in previous case for $\nu A > 1$ we get :

$$\begin{aligned} &KA_1 + LB_1 - q = 0, \\ &KB_1 - LB_1 = 0, \\ &A_0\left(1 + a\left(A_0^2 + \frac{3}{2}A^2\right)\right) + e - h\left(1 + \frac{3}{2}\nu^2 A^2 - b\right) = 0. \end{aligned} \tag{13}$$

Taking into consideration the perturbation solution (8) we have :

$$\begin{aligned} &-2\nu\xi' + M^\circ\xi + N^\circ\eta = 0, \\ &2\nu\eta' + P^\circ\xi + R^\circ\eta = 0, \end{aligned} \tag{14}$$

where :

$$\begin{aligned} &M^\circ = M, \\ &N^\circ = N, \\ &P^\circ = 1 - \nu^2 + 3A_0^2 a + \frac{9}{4}A_1^2 a + \frac{3}{4}B_1^2 a + \frac{3}{2}A_1 B_1 h\nu^3, \\ &R^\circ = \frac{3}{2}A_1 B_1 a - e\nu + 3h\nu + \frac{3}{4}h\nu^3 B_1^2 + \frac{9}{4}h\nu^3 B_1^3. \end{aligned} \tag{15}$$

In this case the necessary conditions for Hopf bifurcation are :

$$2(3h - e) + 3h\nu^2 A^2 = 0, \tag{16}$$

$$P^\circ N^\circ - M^\circ R^\circ > 0. \tag{17}$$

From Eq. (16) we obtain that $e \in (3h,\ 4.5h)$-in further calculations we take $e = 4h$-while from Eq. (13) we get :

$$A_0^3 + \frac{1}{a}\left(1 + \frac{a_2}{\nu}\right)A_0 + 2h - b = 0. \tag{18}$$

This equation has only one real root

$$\begin{aligned} A_0 = &((-g + (g^2 + p^3)^{1/2})^{1/3} + \\ &-((g + (g^2 + p^3)^{1/2})^{1/3}, \end{aligned} \tag{19}$$

where :

$$g = \frac{2h - b}{2a}, \quad p = \frac{1}{3}\left(\frac{1}{a} + \frac{1}{\nu^2}\right). \tag{20}$$

Taking arbitrary the parameters a, b and h for $\nu \in (0.1, 10.)$ we calculate the value

$$q^2 = \frac{2}{\nu^2}\left(\left(1 - \nu^2 + 3aA_0^2 + \frac{3a}{2\nu^2}\right)^2 + \frac{h^2\nu^2}{4}\right), \tag{21}$$

and then verify the condition $\nu A < 1$ and inequality (17). We have found the solution : $\nu = 2.1$, $q^2 = .01$, $a = 10.$, $b = .1$, $h = .1$.

The numerical experiments have been done for the parameters set near these found analytically. In this case, after the Hopf bifurcation of the periodic orbit the regular two frequencies motion has appeared (Fig. 4).

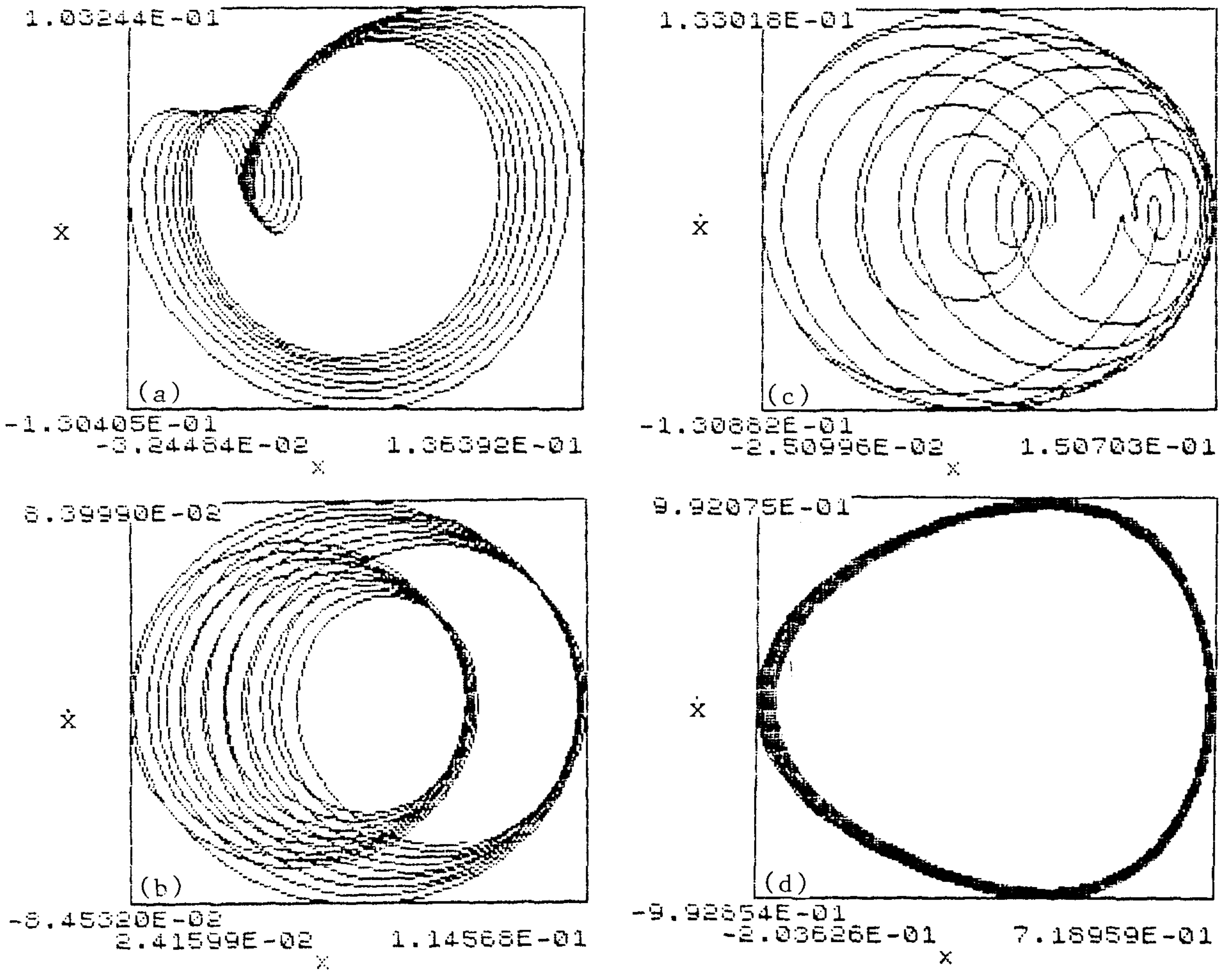

Fig. 4 The evolution of the phase trajectories for $\nu A<1$: $a=10.$, $e=.01$, $h=.001$, $\nu=2.1$, $q=.1$. (a) $b=.05$, (b) $b=.085$, (c) $b=.1$, (d) $b=1.005$.

4. CONCLUDING REMARKS

The paper presents the analytical approach leading to detection of chaotic motion on the example of the nonlinear oscillator with friction. Based on the approximate method, we have chosen such critical parameter sets for which the Hopf bifurcation of the periodic motion with one frequency will occur. Two cases ($\nu A>1$ and $\nu A<1$) were considered. In the first case chaos was found for the parameters nearby critical set, while in the second after Hopf bifurcation of the periodic orbit the new regular motion with two frequencies has appeared.

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